Redefining Mathematical Revolutions

Andrew Aberdein

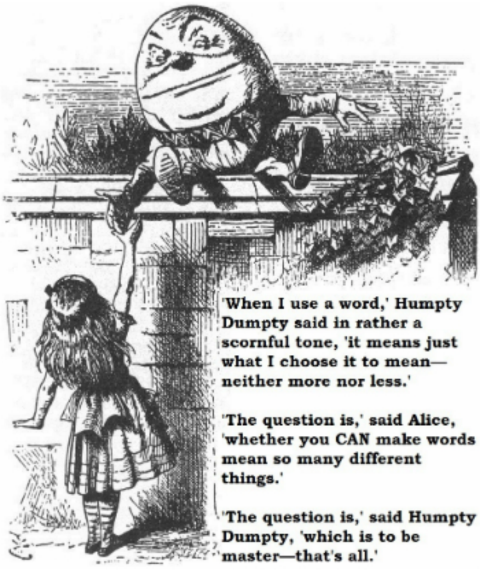
School of Arts and Communication,
Florida Institute of Technology,
150 West University Blvd,
Melbourne, Florida 32901-6975, U.S.A.

aberdein@fit.edu
my.fit.edu/∼aberdein
twitter.com/andrewaberdein
fit.academia.edu/AndrewAberdein

Enabling Mathematical Cultures
Oxford University, December 6, 2017

Outline

There's glory for you!



Lewis Carroll, *Through the Looking Glass*

Revolutions, glorious and otherwise

Glorious revolution: The key components of a theory are preserved, despite changes in their character and relative significance.

Inglorious revolution: Some key component(s) are lost, and perhaps other novel material is introduced by way of replacement.

Paraglorious revolution: All key components are preserved, as in a glorious revolution, but new key components are also added.

Null revolution: No key components change at all.

Andrew Aberdein & Stephen Read, 2009, The philosophy of alternative logics. In Leila Haaparanta, ed., *The Development of Modern Logic*, Oxford University Press

Outline

Incommensurability

Methodological Incommensurability (MI) There are no objective criteria of theory evaluation. The familiar criteria of evaluation, such as simplicity and fruitfulness, are not a fixed set of rules. Rather, they vary with the currently dominant paradigm.

Taxonomic Incommensurability (TI) Periods of scientific change (in particular, revolutionary change) that exhibit TI are scientific developments in which existing concepts are replaced with new concepts that are incompatible with the older concepts. The new concepts are incompatible with the old concepts in the following sense: two competing scientific theories are conceptually incompatible (or incommensurable) just in case they do not share the same "lexical taxonomy". A lexical taxonomy contains the structures and vocabulary that are used to state a theory.

Moti Mizrahi, 2015, Kuhn's incommensurability thesis: What's the argument? *Social Epistemology*, 29(4)

Incommensurability

Methodological Incommensurability (MI) There are no objective criteria of theory evaluation. The familiar criteria of evaluation, such as simplicity and fruitfulness, are not a fixed set of rules. Rather, they vary with the currently dominant paradigm.

Taxonomic Incommensurability (TI) Periods of scientific change (in particular, revolutionary change) that exhibit TI are scientific developments in which existing concepts are replaced with new concepts that are incompatible with the older concepts. The new concepts are incompatible with the old concepts in the following sense: two competing scientific theories are conceptually incompatible (or incommensurable) just in case they do not share the same "lexical taxonomy". A lexical taxonomy contains the structures and vocabulary that are used to state a theory.

Moti Mizrahi, 2015, Kuhn's incommensurability thesis: What's the argument? *Social Epistemology*, 29(4)

Against TI

Why There is No Valid Deductive Support for (TI)

1. Reference change (discontinuity) is conclusive evidence for (TI) only if reference change (discontinuity) entails incompatibility of conceptual content.
2. Reference change (discontinuity) does not entail incompatibility of conceptual content.
3. ∴ It is not the case that reference change (discontinuity) is conclusive evidence for (TI).

Why There is No Strong Inductive Support for (TI)

1. There is a strong inductive argument for (TI) only if there are no rebutting defeaters against (TI).
2. There are rebutting defeaters against (TI).
3. ∴ It is not the case that there is a strong inductive argument for (TI).

Against TI

Why There is No Valid Deductive Support for (TI)

1. Reference change (discontinuity) is conclusive evidence for (TI) only if reference change (discontinuity) entails incompatibility of conceptual content.
2. Reference change (discontinuity) does not entail incompatibility of conceptual content.
3. ∴ It is not the case that reference change (discontinuity) is conclusive evidence for (TI).

Why There is No Strong Inductive Support for (TI)

1. There is a strong inductive argument for (TI) only if there are no rebutting defeaters against (TI).
2. There are rebutting defeaters against (TI).
3. ∴ It is not the case that there is a strong inductive argument for (TI).

Moti Mizrahi, 2015, Kuhn's incommensurability thesis: What's the argument? *Social Epistemology*, 29(4)

Outline

A trilemma?

A trilemma

0. Empirical science and mathematics are methodologically continuous (Aberdein & Mizrahi);
1. Mathematics can exhibit inglorious revolutions (Aberdein);
2. Inglorious revolutions are not characteristic of science (Mizrahi).

Some possible resolutions

$\neg 0 + 1 + 2$: Aberdein and Mizrahi both right about revolutions.

$0 + \neg 1 + 2$: Mizrahi right about revolutions; Aberdein wrong.

$0 + 1 + \neg 2$: Aberdein right about revolutions; Mizrahi wrong.

$\neg 0 + \neg 1 + \neg 2$: Aberdein and Mizrahi both wrong about revolutions.

A trilemma?

A trilemma

0. Empirical science and mathematics are methodologically continuous (Aberdein & Mizrahi);
1. Mathematics can exhibit inglorious revolutions (Aberdein);
2. Inglorious revolutions are not characteristic of science (Mizrahi).

Some possible resolutions

$\neg 0 + 1 + 2$: Aberdein and Mizrahi both right about revolutions.

$0 + \neg 1 + 2$: Mizrahi right about revolutions; Aberdein wrong.

$0 + 1 + \neg 2$: Aberdein right about revolutions; Mizrahi wrong.

$\neg 0 + \neg 1 + \neg 2$: Aberdein and Mizrahi both wrong about revolutions.

A trilemma?

A trilemma

0. Empirical science and mathematics are methodologically continuous (Aberdein & Mizrahi);
1. Mathematics can exhibit inglorious revolutions (Aberdein);
2. Inglorious revolutions are not characteristic of science (Mizrahi).

Some possible resolutions

$\neg 0 + 1 + 2$: Aberdein and Mizrahi both right about revolutions.

$0 + \neg 1 + 2$: Mizrahi right about revolutions; Aberdein wrong.

$0 + 1 + \neg 2$: Aberdein right about revolutions; Mizrahi wrong.

$\neg 0 + \neg 1 + \neg 2$: Aberdein and Mizrahi both wrong about revolutions.

A trilemma?

A trilemma

0. Empirical science and mathematics are methodologically continuous (Aberdein & Mizrahi);
1. Mathematics can exhibit inglorious revolutions (Aberdein);
2. Inglorious revolutions are not characteristic of science (Mizrahi).

Some possible resolutions

$\neg 0 + 1 + 2$: Aberdein and Mizrahi both right about revolutions.

$0 + \neg 1 + 2$: Mizrahi right about revolutions; Aberdein wrong.

$0 + 1 + \neg 2$: Aberdein right about revolutions; Mizrahi wrong.

$\neg 0 + \neg 1 + \neg 2$: Aberdein and Mizrahi both wrong about revolutions.

A trilemma?

A trilemma

0. Empirical science and mathematics are methodologically continuous (Aberdein & Mizrahi);
1. Mathematics can exhibit inglorious revolutions (Aberdein);
2. Inglorious revolutions are not characteristic of science (Mizrahi).

Some possible resolutions

$\neg 0 + 1 + 2$: Aberdein and Mizrahi both right about revolutions.

$0 + \neg 1 + 2$: Mizrahi right about revolutions; Aberdein wrong.

$0 + 1 + \neg 2$: Aberdein right about revolutions; Mizrahi wrong.

$\neg 0 + \neg 1 + \neg 2$: Aberdein and Mizrahi both wrong about revolutions.

A trilemma?

A trilemma

0. Empirical science and mathematics are methodologically continuous (Aberdein & Mizrahi);
1. Mathematics can exhibit inglorious revolutions (Aberdein);
2. Inglorious revolutions are not characteristic of science (Mizrahi).

Some possible resolutions

$\neg 0 + 1 + 2$: Aberdein and Mizrahi both right about revolutions.

$0 + \neg 1 + 2$: Mizrahi right about revolutions; Aberdein wrong.

$0 + 1 + \neg 2$: Aberdein right about revolutions; Mizrahi wrong.

$\neg 0 + \neg 1 + \neg 2$: Aberdein and Mizrahi both wrong about revolutions.

A trilemma?

A trilemma

0. Empirical science and mathematics are methodologically continuous (Aberdein & Mizrahi);
1. Mathematics can exhibit inglorious revolutions (Aberdein);
2. Inglorious revolutions are not characteristic of science (Mizrahi).

Some possible resolutions

$\neg 0 + 1 + 2$: Aberdein and Mizrahi both right about revolutions.

$0 + \neg 1 + 2$: Mizrahi right about revolutions; Aberdein wrong.

$0 + 1 + \neg 2$: Aberdein right about revolutions; Mizrahi wrong.

$\neg 0 + \neg 1 + \neg 2$: Aberdein and Mizrahi both wrong about revolutions.

A trilemma?

A trilemma

0. Empirical science and mathematics are methodologically continuous (Aberdein & Mizrahi);
1. Mathematics can exhibit inglorious revolutions (Aberdein);
2. Inglorious revolutions are not characteristic of science (Mizrahi).

Some possible resolutions

- $\neg 0 + 1 + 2$: Aberdein and Mizrahi both right about revolutions.
- $0 + \neg 1 + 2$: Mizrahi right about revolutions; Aberdein wrong.
- $0 + 1 + \neg 2$: Aberdein right about revolutions; Mizrahi wrong.
- $\neg 0 + \neg 1 + \neg 2$: Aberdein and Mizrahi both wrong about revolutions.

Outline

Irony for mathematicians

. . . a suggestion was made that proofs by contradiction are the mathematician's version of irony. I'm not sure that I agree with that: when we give a proof by contradiction, we make it very clear that we are discussing a counterfactual, so our words *are* intended to be taken at face value

Timothy Gowers, 2012, Vividness in mathematics and narrative.
In Apostolos Doxiadis & Barry Mazur, edd., *Circles Disturbed: The Interplay of Mathematics and Narrative*, Princeton University Press

Three grades of mathematical irony

When might a mathematician, M, assert that p where p is false?

1. Proofs by contradiction: M justifiedly believes $\neg p$;
2. Quandaries: M is justified in believing neither p nor $\neg p$;
3. False conjectures: M justifiedly believes p.

Irony for mathematicians

. . . a suggestion was made that proofs by contradiction are the mathematician's version of irony. I'm not sure that I agree with that: when we give a proof by contradiction, we make it very clear that we are discussing a counterfactual, so our words *are* intended to be taken at face value

Timothy Gowers, 2012, Vividness in mathematics and narrative.
In Apostolos Doxiadis & Barry Mazur, edd., *Circles Disturbed: The Interplay of Mathematics and Narrative*, Princeton University Press

Three grades of mathematical irony

When might a mathematician, M, assert that p where p is false?

1. Proofs by contradiction: M justifiedly believes $\neg p$;
2. Quandaries: M is justified in believing neither p nor $\neg p$;
3. False conjectures: M justifiedly believes p.

Irony for mathematicians

. . . a suggestion was made that proofs by contradiction are the mathematician's version of irony. I'm not sure that I agree with that: when we give a proof by contradiction, we make it very clear that we are discussing a counterfactual, so our words *are* intended to be taken at face value

Timothy Gowers, 2012, Vividness in mathematics and narrative.
In Apostolos Doxiadis & Barry Mazur, edd., *Circles Disturbed: The Interplay of Mathematics and Narrative*, Princeton University Press

Three grades of mathematical irony

When might a mathematician, M, assert that p where p is false?

1. Proofs by contradiction: M justifiedly believes $\neg p$;
2. Quandaries: M is justified in believing neither p nor $\neg p$;
3. False conjectures: M justifiedly believes p.

Irony for mathematicians

... a suggestion was made that proofs by contradiction are the mathematician's version of irony. I'm not sure that I agree with that: when we give a proof by contradiction, we make it very clear that we are discussing a counterfactual, so our words *are* intended to be taken at face value

Timothy Gowers, 2012, Vividness in mathematics and narrative.
In Apostolos Doxiadis & Barry Mazur, edd., *Circles Disturbed: The Interplay of Mathematics and Narrative*, Princeton University Press

Three grades of mathematical irony

When might a mathematician, M, assert that p where p is false?

1. Proofs by contradiction: M justifiedly believes $\neg p$;
2. Quandaries: M is justified in believing neither p nor $\neg p$;
3. False conjectures: M justifiedly believes p.

Outline

The World Without End Hypothesis

a hypothesis based on the existence of the θ_j ... (called "the world without end hypothesis" by the authors here) was so compelling that many believed the θ_j must exist; now that we know they don't, the behavior of the *EHP* sequence is much more mysterious

Paul G. Goerss, 2011, Review of Hill, Michael A.; Hopkins, Michael J.; Ravenel, Douglas C. 'The Arf-Kervaire problem in algebraic topology: sketch of the proof'. *Mathematical Reviews*, MR2906370.

Researchers developed what Ravenel calls an entire "cosmology" of conjectures based on the assumption that manifolds with Arf-Kervaire invariant equal to 1 exist in all dimensions of the form $2n - 2$

Erica Klarreich, 2009, Mathematicians solve 45-year-old Kervaire invariant puzzle, *Simons Science News*

In the light of the above conjecture and the failure over fifty years to construct framed manifolds of Arf-Kervaire invariant one this might turn out to be a book about things which do not exist

Victor P. Snaith, 2009, *Stable Homotopy Around the Arf-Kervaire Invariant*, Birkhäuser

The Doomsday Hypothesis

Main Theorem

The Arf-Kervaire elements $\theta_j \in \pi^S_{2^{j+1}-2}$ do not exist for $j \geq 7$.

The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. Browder's theorem says that such things can exist only in dimensions that are 2 less than a power of 2.

Some homotopy theorists, most notably Mahowald, speculated about what would happen if θ_j existed for all j. They derived numerous consequences about homotopy groups of spheres. The possible nonexistence of the θ_j for large j was known as the Doomsday Hypothesis

Michael A. Hill, Michael J. Hopkins, & Douglas C. Ravenel, 2009, The Arf-Kervaire invariant problem in algebraic topology, *Current Developments in Mathematics*

The Doomsday Hypothesis

Main Theorem

The Arf-Kervaire elements $\theta_j \in \pi^S_{2^{j+1}-2}$ do not exist for $j \geq 7$.

The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. Browder's theorem says that such things can exist only in dimensions that are 2 less than a power of 2.

Some homotopy theorists, most notably Mahowald, speculated about what would happen if θ_j existed for all j. They derived numerous consequences about homotopy groups of spheres. The possible nonexistence of the θ_j for large j was known as the Doomsday Hypothesis

Inter-Universal Teichmüller Theory

abc conjecture (Oesterlé–Masser, 1985)

For every $\varepsilon > 0$, there are only finitely many triples (a, b, c) of coprime positive integers where $a + b = c$, such that $c > d^{1+\varepsilon}$, where d denotes the product of the distinct prime factors of abc.

For example, try $a = 15$ and $b = 28$. These are coprime, but $c = 43$ and $d = 2 \times 3 \times 5 \times 7 \times 43 = 9030 \gg 43$. So $(15, 28, 43)$ is not one of the specified triples (for any ε).
On the other hand, let $a = 1$ and $b = 63$. Then we have $c = 64$ and $d = 2 \times 3 \times 7 = 42 < 64$. So $(1, 63, 64)$ is such a triple (at least for values of $\varepsilon < .11269$).

Inter-Universal Teichmüller Theory

abc conjecture (Oesterlé–Masser, 1985)

For every $\varepsilon > 0$, there are only finitely many triples (a, b, c) of coprime positive integers where $a + b = c$, such that $c > d^{1+\varepsilon}$, where d denotes the product of the distinct prime factors of abc.

For example, try $a = 15$ and $b = 28$. These are coprime, but $c = 43$ and $d = 2 \times 3 \times 5 \times 7 \times 43 = 9030 \gg 43$. So $(15, 28, 43)$ is not one of the specified triples (for any ε).
On the other hand, let $a = 1$ and $b = 63$. Then we have $c = 64$ and $d = 2 \times 3 \times 7 = 42 < 64$. So $(1, 63, 64)$ is such a triple (at least for values of $\varepsilon < .11269$).

Inter-Universal Teichmüller Theory

abc conjecture (Oesterlé–Masser, 1985)

For every $\varepsilon > 0$, there are only finitely many triples (a, b, c) of coprime positive integers where $a + b = c$, such that $c > d^{1+\varepsilon}$, where d denotes the product of the distinct prime factors of abc.

For example, try $a = 15$ and $b = 28$. These are coprime, but $c = 43$ and $d = 2 \times 3 \times 5 \times 7 \times 43 = 9030 \gg 43$. So $(15, 28, 43)$ is not one of the specified triples (for any ε).
On the other hand, let $a = 1$ and $b = 63$. Then we have $c = 64$ and $d = 2 \times 3 \times 7 = 42 < 64$. So $(1, 63, 64)$ is such a triple (at least for values of $\varepsilon < .11269$).

Inter-Universal Teichmüller Theory

But so far, the few who have understood the work have struggled to explain it to anyone else. "Everybody who I'm aware of who's come close to this stuff is quite reasonable, but afterwards they become incapable of communicating it," says one mathematician who did not want his name to be mentioned.

Davide Castelvecchi, 2015, The biggest mystery in mathematics: Shinichi Mochizuki and the impenetrable proof, *Nature*, 526

Mochizuki's four IUT theory papers . . . were the subject of the last two days of the conference. The job of explaining those papers fell to Chung Pang Mok of Purdue University and Yuichiro Hoshi and Go Yamashita, both colleagues of Mochizuki's at the Research Institute for Mathematical Sciences at Kyoto University. The three are among a small handful of people who have devoted intense effort to understanding Mochizuki's IUT theory. By all accounts, their talks were impossible to follow.

Kevin Hartnett, 2015, Hope rekindled for perplexing proof, *Quanta Magazine*

Inter-Universal Teichmüller Theory

But so far, the few who have understood the work have struggled to explain it to anyone else. "Everybody who I'm aware of who's come close to this stuff is quite reasonable, but afterwards they become incapable of communicating it," says one mathematician who did not want his name to be mentioned.

Davide Castelvecchi, 2015, The biggest mystery in mathematics: Shinichi Mochizuki and the impenetrable proof, *Nature*, 526

Mochizuki's four IUT theory papers ... were the subject of the last two days of the conference. The job of explaining those papers fell to Chung Pang Mok of Purdue University and Yuichiro Hoshi and Go Yamashita, both colleagues of Mochizuki's at the Research Institute for Mathematical Sciences at Kyoto University. The three are among a small handful of people who have devoted intense effort to understanding Mochizuki's IUT theory. By all accounts, their talks were impossible to follow.

Kevin Hartnett, 2015, Hope rekindled for perplexing proof, *Quanta Magazine*

Inter-Universal Teichmüller Theory

Typically, when a researcher with a solid track record in mathematical research decides to read a mathematical paper, unlike the case with students or novices who take the time to *study step by step from the rudiments of a subject*, such a researcher will attempt to digest the content of the paper in as efficient a way as is possible, by *scanning* the paper for important terms and theorems so that the researcher may apply his/her vast store of expertise and deep understanding of the subject to determine just *which* of those topics of the subject that, from point of view of the researcher, have already been "digested" and "well understood" *play a key role in the paper*.

Certain researchers believe that every essential phenomenon in number theory may in fact be reduced to some aspect of the *representation-theoretic* approach exemplified by the *Langlands program*. On the other hand, the fundamental ideas of IUTeich are not based on this sort of representation-theoretic approach.

Shinichi Mochizuki, 2014, On the verification of inter-universal Teichmüller theory: A progress report (as of December 2014)

Classical → Modern → Contemporary

Classical mathematics (midseventeenth to midnineteenth centuries):
sophisticated use of the infinite (Pascal, Leibniz, Euler, Gauss);

Modern mathematics (midnineteenth to midtwentieth centuries):
sophisticated use of structural and qualitative properties (Galois, Riemann, Hilbert);

Contemporary mathematics (midtwentieth century to present):
sophisticated use of the properties of transference, reflection and gluing (Grothendieck, Serre, Shelah).

Fernando Zalamea, 2012, *Synthetic Philosophy of Contemporary Mathematics*, Urbanomic

Classical $\to$ Modern $\to$ Contemporary

Classical mathematics (midseventeenth to midnineteenth centuries):
sophisticated use of the infinite (Pascal, Leibniz, Euler, Gauss);

Modern mathematics (midnineteenth to midtwentieth centuries):
sophisticated use of structural and qualitative properties (Galois, Riemann, Hilbert);

Contemporary mathematics (midtwentieth century to present):
sophisticated use of the properties of transference, reflection and gluing (Grothendieck, Serre, Shelah).

Fernando Zalamea, 2012, *Synthetic Philosophy of Contemporary Mathematics*, Urbanomic

Classical → Modern → Contemporary

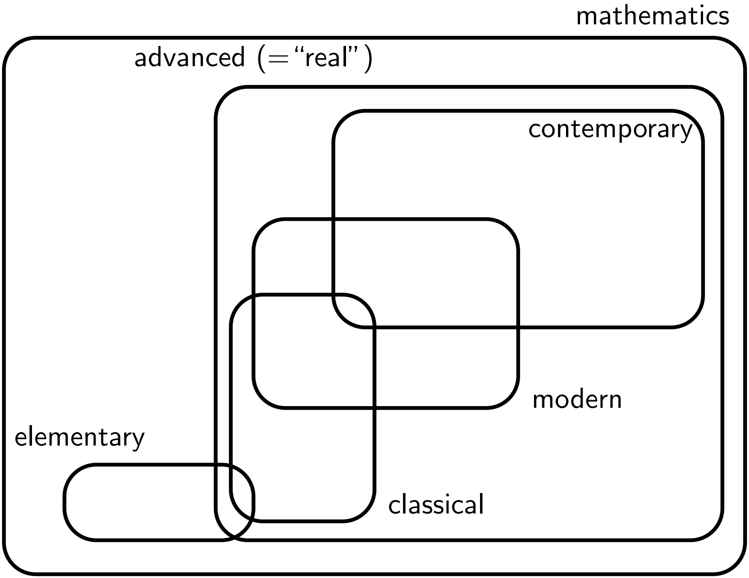
Classical mathematics (midseventeenth to midnineteenth centuries):
sophisticated use of the infinite (Pascal, Leibniz, Euler, Gauss);

Modern mathematics (midnineteenth to midtwentieth centuries):
sophisticated use of structural and qualitative properties (Galois, Riemann, Hilbert);

Contemporary mathematics (midtwentieth century to present):
sophisticated use of the properties of transference, reflection and gluing (Grothendieck, Serre, Shelah).

Fernando Zalamea, 2012, *Synthetic Philosophy of Contemporary Mathematics*, Urbanomic

Classical → Modern → Contemporary



Fernando Zalamea, 2012, *Synthetic Philosophy of Contemporary Mathematics*, Urbanomic

Outline

There's glory for you!

Mizrahi against incommensurability

A trilemma?

Irony for mathematicians

Examples
- World Without End → Doomsday
- Inter-Universal Teichmüller Theory
- Classical → Modern → Contemporary

Conclusion

Conclusion

Some prospects for mathematical revolution:

- TI through (exposure of) false conjectures?
- TI through paraglorious revolution?
- TI through non-transitive sequence of glorious revolutions?

Where does this leave us? Several possibilities:

- Mizrahi's argument works against mathematical as well as scientific revolutions.
- Mizrahi's argument works against neither scientific or mathematical revolutions.
- Mizrahi's argument works only against scientific revolutions. So only mathematics has "true" revolutions!

Conclusion

Some prospects for mathematical revolution:

- TI through (exposure of) false conjectures?
- TI through paraglorious revolution?
- TI through non-transitive sequence of glorious revolutions?

Where does this leave us? Several possibilities:

- Mizrahi's argument works against mathematical as well as scientific revolutions.
- Mizrahi's argument works against neither scientific or mathematical revolutions.
- Mizrahi's argument works only against scientific revolutions. So only mathematics has "true" revolutions!

Conclusion

Some prospects for mathematical revolution:

- TI through (exposure of) false conjectures?
- TI through paraglorious revolution?
- TI through non-transitive sequence of glorious revolutions?

Where does this leave us? Several possibilities:

- Mizrahi's argument works against mathematical as well as scientific revolutions.
- Mizrahi's argument works against neither scientific or mathematical revolutions.
- Mizrahi's argument works only against scientific revolutions. So only mathematics has "true" revolutions!

Conclusion

Some prospects for mathematical revolution:

- TI through (exposure of) false conjectures?
- TI through paraglorious revolution?
- TI through non-transitive sequence of glorious revolutions?

Where does this leave us? Several possibilities:

- Mizrahi's argument works against mathematical as well as scientific revolutions.
- Mizrahi's argument works against neither scientific or mathematical revolutions.
- Mizrahi's argument works only against scientific revolutions. So only mathematics has "true" revolutions!

Conclusion

Some prospects for mathematical revolution:

- TI through (exposure of) false conjectures?
- TI through paraglorious revolution?
- TI through non-transitive sequence of glorious revolutions?

Where does this leave us? Several possibilities:

- Mizrahi's argument works against mathematical as well as scientific revolutions.
- Mizrahi's argument works against neither scientific or mathematical revolutions.
- Mizrahi's argument works only against scientific revolutions. So only mathematics has "true" revolutions!

Conclusion

Some prospects for mathematical revolution:

- TI through (exposure of) false conjectures?
- TI through paraglorious revolution?
- TI through non-transitive sequence of glorious revolutions?

Where does this leave us? Several possibilities:

- Mizrahi's argument works against mathematical as well as scientific revolutions.
- Mizrahi's argument works against neither scientific or mathematical revolutions.
- Mizrahi's argument works only against scientific revolutions. So only mathematics has "true" revolutions!

Conclusion

Some prospects for mathematical revolution:

- TI through (exposure of) false conjectures?
- TI through paraglorious revolution?
- TI through non-transitive sequence of glorious revolutions?

Where does this leave us? Several possibilities:

- Mizrahi's argument works against mathematical as well as scientific revolutions.
- Mizrahi's argument works against neither scientific or mathematical revolutions.
- Mizrahi's argument works only against scientific revolutions. So only mathematics has "true" revolutions!